tion are given. A vertical plane passed through the axis of the road divides it into two solids, such as we have been considering. It can, therefore, be correctly calculated as one prismoid.

The rule may also be applied to "*irregular cross-sections*," or those in which the surface of the ground is so irregular that a number of levels have to be taken at various points of each profile. To do this, we must conceive vertical planes to pass through all the points of each profile at which the transverse slope of the ground changes. They will thus divide the solid into a number of solids with warped surfaces, such as we have been considering. The principle of our third hypothesis will then enable us to determine by a simple proportion the height at which the vertical plane passing through any angular summit of one profile cuts the top line of the other. This will furnish the data for applying the preceding rule.

If the views here presented should meet with general acceptance, engineers would be enabled to economize much time and labor, since they would no longer feel compelled, as now, to take their cross-sections so near together that the ground between them should be approximately plane, but could take them as far apart as the ground *varied uniformly*, no matter how much, or for what distance, that might be.

The adoption of these principles would also seriously modify the methods now employed in the calculation of earthwork, and the results obtained by them, particularly the use of "equivalent mean heights." The numerical details of these professional matters would, however, be out of place here.

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4. On the Law of Human Mortality that appears to obtain in Massachusetts, with Tables of practical value deduced therefrom. By E. B. Elliott, of Boston.

The accompanying tables comprise part of an original series, that has been prepared for the New England Mutual Life Insurance Company of Boston, from extensive and reliable European and American data. They have been calculated from official abstracts of observations

made for the year 1855, respecting the status and the movements of the population in the 166 of the 331 towns of the Commonwealth of Massachusetts, in each of which, the ratio of the number of registered deaths, to the number of the population, was *greater* than one to sixty-three.

With the numbers returned for the 166 towns, have been included two thirds of the numbers of the population, births, and deaths of the three State almshouses; the population of the 166 towns being two thirds (.663) of the population of the entire State.

The aggregate population of these communities, as returned for the first day of June, 1855, was 751,241, and the registered deaths in these towns during the year was 16,086. The well-known Carlisle table of mortality was deduced from only 1,840 deaths, registered during the nine years 1779–87, the mean population of the period being 8,177.

The rate of mortality, or the ratio of deaths to the population, according to the returns, was probably somewhat lower, in these communities, than the rate that actually prevailed in them, in consequence of probable omissions in the registration of deaths in some of the districts.

But these communities embrace a larger proportion of the more populous districts of the State, and of those in which we should expect the prevailing rate of mortality to be higher than the rate for the entire State; and we are probably safe in concluding that the law of mortality obtaining in these districts, according to the returns, does not greatly vary from the law of mortality actually prevailing over the entire population of the Commonwealth.

It is not possible, supplied with only our present information, to indicate the precise line of separation between the reliable and the questionable data; but it is thought that such a division has been made that the data retained are affected by only inconsiderable errors, and the errors of excess and of defect nearly compensate for each other.

The tables deduced from the resulting law of mortality are intended to facilitate the solution of certain problems in political arithmetic, and to furnish replies to questions involving the probable duration of human life.

Tables are also given, comparing the rates of mortality obtaining in Massachusetts at divers intervals of age, with corresponding rates obtaining in certain European communities.

The "Fourteenth Annual Report relating to the Registry and Returns of Births, Marriages, and Deaths, in Massachusetts, for the Year 1855," and the "Abstract of the Census of the Commonwealth of Massachusetts, taken with reference to facts existing on the first day of June, 1855," prepared under the direction of the Hon. Francis DeWitt, Secretary of the Commonwealth, and under the supervision of Dr. Nathaniel B. Shurtleff, of Boston, appear to have been the first and the only official reports, whether State or national, in which either the ages of the persons living, or the ages of those dying, in the State, have been *distinguished by towns;* consequently, they are the first that furnish data, from comparison of which it has been possible to construct a Life Table that can satisfactorily express the law of mortality prevailing over any considerable portion of the people of the Commonwealth.* † In previous reports the ages have been distinguished only *by counties.*

Since the Registration Act of 1849, the registrars, in many of the towns of the Commonwealth, appear to have annually furnished the office of Secretary of State with very valuable and accurate statistics respecting the births, marriages, and deaths occurring in their respective districts.

In other towns the results indicate that this information has been but imperfectly recorded, while in a few cases the officers have uniformly neglected either to register or report.

In registration reports, previous to that for the year 1855, the full

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* A Life Table, prepared by the eminent Dr. Edward Wigglesworth from sixty-two Bills of Mortality recorded previous to the year 1789 in the States of Massachusetts and of New Hampshire, was published in the second volume of the Memoirs of the American Academy of Arts and Sciences. Unfortunately, in constructing the table, allowance was not made for the fact that the population had been rapidly increasing, and the table was framed on the assumption that it had been stationary. This table has been employed by the courts of the Commonwealth in determining the values of "life interests in estates, legacies, and pensions," and the values of "reversions in heritable property."

† A valuable Life Table, constructed from observations respecting the mortality of the Alumni of Harvard University, and, consequently, expressing the law of mortality which prevails, in America, over the more highly educated classes, was laid before the American Association at its late meeting in Montreal, by Prof. Benjamin Peirce, of Cambridge.

and reliable information respecting the ages of the dying furnished by the able and competent registrars, in certain towns, has been vitiated by union with the questionable or obviously defective data obtained from other towns in the same county.

In the Registration Report for 1855, the table which distinguishes by age, sex, and locality the deaths registered during the year (Table VI.), was prepared with special reference to its employment in the construction of Life Tables, and is believed to be a faithful abstract from the returns.

The enumeration of the numbers and ages of the population of Massachusetts, according to the State Census of 1855, and the previous enumerations ordered by the General Government, may safely be regarded as reliable.

It is to be regretted that the abstracts of the Census for 1855, in giving the ages of the population, did not distinguish the *sexes*.

This deficiency may, in a measure, be supplied by assuming that the proportional distribution of the sexes at the different ages in 1855, was the same as that obtaining in Massachusetts in 1850, according to the National Census of that year.

Of the 324 statistical districts into which England and Wales are subdivided, in only *two* was there indicated, according to very accurate observations for the seven years, 1838–44, an annual rate of mortality *less* than one death to sixty-three persons living. Of the 331 towns in Massachusetts (nearly the same in number with the English districts just mentioned), there were, in 1855, 165 towns, in each of which the rate of mortality, indicated by the returns, was *less* than one to sixty-three. The returns from the three State almshouses are not included with the returns of the towns in which they happen to be located, namely, Monson, Tewksbury, and Bridgewater. The average population of the 331 towns of Massachusetts is much less than that of the 324 English districts; the population of Massachusetts, in 1855, being one million (1,132,369), and that of England and Wales, in 1841, sixteen millions (15,927,867). The above assumed test, in selecting returns showing a mortality over one to sixty-three, is accordingly confirmed by the English returns, which are made, as is well known, with great accuracy.

The mortality of the 324 English districts varied from 1 in 70 to 1

in 30; in the 331 towns of Massachusetts, according to the returns, from *zero* to 1 in 30. The mortality of Massachusetts, according to the entire returns, 1.84 per cent., or 1 in 54; in the 166 towns it was 2.14 per cent., or 1 in 47.

AVERAGE ANNUAL RATES OF MORTALITY OBTAINING IN THE WHOLE, OR IN PARTS OF SEVEN COUNTRIES OF EUROPE, ARRANGED IN THE INVERSE ORDER OF THEIR RESPECTIVE INTENSITIES.

Years in which the Deaths occurred.	Countries.	To 100 Persons Living,	One Death
		Deaths.	To Pers'ns Living.
7 years 1838–44	England and Wales *	2.19	46.
20 " 1821–40	Sweden †	2.34	43.
4 " 1839–42	France ‡	2.36	42.
9 " 1842–50	Belgium §	2.42	41.
3 " 1839–41	Prussia ‡	2.70	37.
3 " 1834, 7, 9	Austria ‡	3.09	32.
1 " 1842	Russia ‡	3.59	28.

According to the above, it appears that in England and Wales the conditions are most favorable to vitality, and in Russia, least; the several countries, arranged in the inverse order of the respective intensities of mortality, being England, Sweden, France, Belgium, Prussia, Austria, and Russia.

The population of Massachusetts has increased more rapidly than those of the principal countries of Europe. The annual rate of increase of the former, during the years 1850–55, was two and two thirds per cent. (2.63). The effect of an increase by births merely is to diminish, in some degree, the general rate of mortality, even though the intensity of mortality, at different ages, remains unchanged.

We will now proceed to the construction of the Life Table.

* 9th Report Registrar General (England).

† 8th Report Registrar General (England).

‡ See 6th Report Registrar General (England).

§ Statistique de la Belgique, Publié par le Ministre de l'Interieur.

TABLE I.—NUMBER AND AGES OF THE POPULATION OF MASSACHUSETTS, ACCORDING TO THE NATIONAL CENSUS OF 1850 AND OF THE STATE CENSUS OF 1855; ALSO THE ANNUAL RATIO OF INCREASE, AND THE LOGARITHMS OF THE MONTHLY RATIO OF INCREASE, THE RATIOS BEING CORRECTED FOR THE NUMBERS RETURNED AT AGES NOT SPECIFIED.

Ages.	Population.		Unity plus the Annual Rate of Increase.	Logarithms of the Monthly Ratio of Increase.
	1850.	1855.		
0–1	23,192			
1–5	90,853			
0–5	114,045	132,944	1.03131	.0011159
5–10	102,797	115,862	1.02439	.0008720
10–15	98,024	110,098	1.02367	.0008468
15–20	105,741	117,047	1.02069	.0007413
20–30	210,997	235,678	1.02254	.0008067
30–40	143,931	165,046	1.02793	.0009968
40–50	96,266	111,500	1.02999	.0010694
50–60	60,254	71,829	1.03594	.0012779
60–70	36,837	42,423	1.02881	.0010279
70–80	17,936	20,810	1.02034	.0010818
80–90	5,820	6,138	1.01086	.0003911
90–100	613	634	1.00693	.0002498
100 and upwards	19	19	1.00017	.0000060
Age not specified	1,234	2,341		
All ages	994,514	1,132,369	1.02630	.0009396
Specified ages	993,280	1,130,028		
90 and upwards	632	653		.0002426

From the returns of births for the six years, 1850–55, it appears that the annual rate of increase was 3.49 per cent. But the annual rate of the increase of the living, under the age of five years, was only 3.13. The latter is believed to be the more correct. The difference is probably owing to a gradual improvement in the completeness of the returns of the births.

In that which follows, we shall assume that the population in the selected districts increased, at the different intervals of age, at the same rates as in the entire State.

The population of the 166 towns may be divided into two classes, the *migratory* and the *permanent;* the former comprising immigrants and emigrants.

We assume that the proportional distribution of the ages of those living under the age of *five* years, was the same in the latter as in the former class; that the ratio of the number of births to the number living under age five, was the same in each; and that the same *invariable* law of mortality prevailed over those under the age of five years in each.

In the towns selected, the number of births registered in 1855 was 23,481. The number of persons living under age five, estimated with reference to the middle of the year and corrected for those returned at unspecified ages, was 90,260.

	Ages.	Deaths.
The deaths at different ages under five years, corrected for those returned at unspecified ages, were	0–1	3,622
	1–2	1,654
	2–3	705
	3–4	375
	4–5	252

Assuming correctness of the returns upon which the above values depend, we find the annual rate of increase of births in the *permanent* population to have been 1.1023 per cent.* We also find the number of

* Had the rate of the annual increase of the numbers living under age five (3.13 per cent.) resulted entirely from the increase of births in a permanent population, the number of births of 1855 would have been 24,457, instead of 23,481, the number registered. On the other hand, had the increase resulted wholly from migration (the annual number of births in the permanent population being constant), the number of

deaths at the different ages demanded by a constant supply of 23,481 annual births in a community influenced by migration, but subject to the above-mentioned invariable law of mortality, to be

Ages.	Deaths.
0–1	3,641.9
1–2	1,681.4
2–3	724.6
3–4	389.7
4–5	264.7

Hence of 10,000 persons born alive, there would survive, ages 1, 2, 3, 4, and 5, as follows : —

TABLE II.

Ages.	Numbers.	Logarithms.
0	10,000.0	4.0000000
1	8,449.0	3.9268053
2	7,732.9	3.8883424
3	7,424.3	3.8706555
4	7,258.3	3.8608349
5	7,145.6	3.8540387

To continue the above table, we shall need to compare the deaths and the population at the different intervals of age.

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births would have been only 22,956. The number of births registered is somewhat nearer the latter than the former of these two values.

Assuming the correctness of the returns of births, deaths, and population in the selected districts, and of the indicated rates of increase of population, it appears that 35 per cent. of the increase of the population *under age five* was due to increase of births in the permanent portion of the population, and 65 per cent. due to the movement of the migratory portion; also that 38 per cent. of the increase of population *at all ages* was due to excess of births over deaths, leaving 62 per cent. to be accounted for by excess of immigration over emigration. (A.)

TABLE III. — DEATHS, POPULATION, MORTALITY, AND LOGARITHMS OF THE PROBABILITY OF LIVING.

Massachusetts, 166 *towns*, 1855.

AGES.	REGISTERED DEATHS. 1855.	POPULATION. June 1st, 1855.	MORTALITY. Ratio of the Number of Deaths in 1855, to the Number of the Population, estimated for the middle of that year, and corrected for the Numbers at ages not specified.	LOGARITHMS, WITH THE SIGN CHANG'D, OF THE PROBABILITY OF SURVIVING EACH INTERVAL OF AGE. Duplicate Values, each deduced from *two* consecutive Ratios in the Column of Mortality.	Arithmetrical Mean of the Duplicate Values.
Under 1 year . .	3,595				
1–2	1,642				
2–3	700	89,852	.0732094		
3–4	372				
4–5	250				
3–5	622		.0191772*		
5–10	595	76,566	.0077980	.0168125 .0169100	.0168613
10–15	311	71,851	.0043436	.0094179 .0094488	.0094333
15–20	672	76,854	.0087768	.0190957 .0190705	.0190831
20–30	1,817	161,544	.0112883	.0491767 .0490648	.0491208
30–40	1,388	112,489	.0123781	.0538119 .0538257	.0538188
40–50	1,035	73,604	.0141040	.0613538 .0615053	.0614295
50–60	908	45,134	.0201687	.0881465 .0884615	.0883040
60–70	942	25,766	.0366732	.1624950 .1624646	.1624798
70–80	976	12,265	.0798130	.3729434 .3595270	.3684713†
80–90	635	3,469	.1838871	1.1507632 .3622095	.9536248†
90 and upwards . .	129	374	.3466159		
Age not specified .	119	1,472			
All ages	16,086	751,240	.0213663		
Specified ages . .	15,967	749,768			

* From deaths and *estimated* population, at the ages of three to five.

† The former of these values was obtained by giving *double*, and the latter by giving *triple, weight* to the antecedent of the respective duplicate values in the preceding column.

The method adopted in calculating the probabilities of living from the annual rate of mortality, is essentially the same as that indicated on pages 60 and 61 of the Proceedings of the American Association for 1856.

From the above Tables II. and III. the following values, from birth to age 90 inclusive, are readily deduced.

TABLE IV. — PROPORTIONS BORN ALIVE, AND SURVIVING CERTAIN AGES.

Ages.	Logarithms.	Numbers.
0	4.0000000	10,000
1	3.9268053	8,449
2	3.8883424	7,733
3	3.8706555	7,424
4	3.8608349	7,258
5	3.8540387	7,146
10	3.8371774	6,873
15	3.8277441	6,726
20	3.8086610	6,437
30	3.7595402	5,748
40	3.7057214	5,078
50	3.6442919	4,409
60	3.5559879	3,597
70	3.3935081	2,475
80	3.0250368	1,059
90	2.0714120	117.9
100	.3430527	2.20

In assigning the average number that may be expected to survive age 100 out of a stated number of births, there is room for some diversity of opinion. The influence, however, of the numbers at this extreme age upon tables of practical utility is inconsiderable.

The logarithms in Table V. were derived from those in Table IV., by the interpolation of eight additional values, namely, those at the ages of 25, 35, 45, 55, 65, 75, 85, and 95. The third differences of the logarithms from age 35 upwards in the following table constitute a constantly increasing series.

Table V.—Massachusetts Life Table, 1855.*

	The number of persons *living at certain ages, to 10,000 children born alive.* Also, the annual number of *deaths at and over certain ages, in a stationary population* supplied by 10,000 annual births.		The *years* which the (L_x) persons *living at certain ages will live*; also, the *years* which those annually *dying at and over* certain ages, in the stationary population, *have lived over those ages*; persons *living at and over certain ages*, in the stationary population.	*Years* which the (Q_x) pesons *living at and over certain ages*, in the stationary population, *will live*; also, the *years* which they *have lived* over those ages.	Average number of *years* which those *living* at certain ages *will live*; also, which those *dying, at and over* certain ages, *have lived* over those ages.	*Average* number of *years* which those *living at and over certain ages will live*; also, which *they have lived* over those ages.
Ages	Logarithms.	Numbers.	$dx\int_x^\infty L_x.$	$dx\int_x^\infty Q_x=(dx)^2\int_x^\infty\int_x^\infty L_x.$	$\frac{Q_x}{L_x}.$	$\frac{Y_x}{Q}.$
x	λL_x	L_x	Q_x	Y_x	E_x	E'_x
0	4.000000	10,000	397,653	12,857,379	39.77	32.33
1	3.926805	8,449	388,655		46.00	
2	3.888342	7,733	380,616		49.22	
3	3.870656	7,424	373,060		50.25	
4	3.860835	7,258	365,727		50.39	
5	3.854039	7,146	358,530	10,967,776	50.17	30.59
10	3.837177	6,873	323,508	9,263,748	47.07	28.64
15	3.827744	6,726	289,508	7,731,641	43.04	26.71
20	3.808661	6,437	256,561	6,367,019	39.86	24.82
25	3.785307	6,100	225,205	5,163,297	36.92	22.93
30	3.759540	5,748	195,584	4,112,047	34.03	21.02
35	3.733016	5,408	167,699	3,204,550	31.01	19.11
40	3.705721	5,078	141,486	2,432,279	27.86	17.19
45	3.676505	4,748	116,919	1,786,956	24.62	15.28
50	3.644292	4,409	94,015	1,260,344	21.32	13.41
55	3.604475	4,022	72,919	843,806	18.13	11.57
60	3.555988	3,597	53,842	527,821	14.97	9.80
65	3.486461	3,065	37,152	301,421	12.12	8.11
70	3.393508	2,475	23,279	151,568	9.41	6.51
75	3.263163	1,833	12,471	63,578	6.80	5.10
80	3.025037	1,059	5,245.2	20,780	4.95	3.96
85	2.640633	437.1	1,599.7	5,075.5	3.66	3.17
90	2.071412	117.9	336.8	869.0	2.86	2.58
95	1.311480	20.49	46.4	94.8	2.26	2.04
100	0.343053	2.203	3.47	5.12	1.58	1.48

* This comprehensive form was first given to the Life Table by Dr. Farr, the eminent English statistician. Valuable details respecting the properties and uses of the columns Q and Y (Table V.) and Z (Table VII.) may be found in the Sixth Report of the Registrar-General (Eng.)

The integration of the functions L_x, Q_x, L'_x, and Q'_x, to obtain the values in columns Q, Y, N, and Z respectively, was chiefly effected by the brief methods detailed in the Proceedings of the American Association for 1856. The ordinary process involves a preliminary and formidable interpolation of values at annual intervals of age, and a summation of the values thus obtained. In note B is offered a modification of the method previously given, especially adapted to the computation of values at the higher ages.

A large variety of useful problems may be solved by reference to the table above. We can now only advert to some of the more obvious of its properties.

According to the law of mortality for Massachusetts, it appears, that of 10,000 children born alive, 6,437 persons *will survive* age 20;

That these 6,437 persons *will live*, in the aggregate, 256,651 years;

That the *average* number of years which they will live is 39.86;

And that the *average* number of years which they *have lived* and *will live*, that is, the complete average duration of life, past and future, in years, is 59.86, that is, 20 + 39.86.

In a *stationary population*, supplied by 10,000 annual births, there will annually occur 6,437 *deaths* of persons at and over age 20.

These 6,437 persons *dying* will have lived, in the aggregate, 256,561 years over age 20.

The *average* number of years over age 20 which they will have lived is 39.86;

Their average age at death is consequently (20 + 39.86 =) 59.86 years.

In a *stationary population* supplied by 10,000 annual births, there will be 256,561 persons constantly living at and over age 20.

This generation of 256,561 persons *will live* in the aggregate 6,367,019 years;

They *have already lived* 6,367,019 years over age 20.

The *average* number of years which they *will live* is 24.82.

The *average* number which they *have lived*, over age 20, is 24.82 years; their average age is consequently 44.82 years; and the complete *average* duration, past and future, of the generation of persons now at and over 20 years of age, or their average age at death, is (44.82 + 24.82 =) 69.64 years.

In a stationary population there constantly will be *living*, to one annual death, 39.86 persons, at and over age 20.

In a community the members of which enter in constant and uniform numbers at age 20, and retire at age 60 or before in the event of death, the average number of years that the present members *will continue* with the community is 18.18; they *have* already *been* members 18.18 years; consequently their complete average duration of membership, past and future, is 36.36 years.

According to the English Life Table (1841) these numbers would be 18.23 and 36.46, respectively.

This case approximately represents that of a community of business men, if we assume that its members enter at about the age of 20 years in nearly equal annual numbers, and retire from active life about the age of 60 years, or before in case of decease.

This table will be found of practical utility, not only for the very valuable purposes of Life Insurance, but also to the statesman and to the political economist, in the solution of many important problems, among which may be mentioned those relating to the strength and the decadence of armies in time of peace, and to the influence of immigration and emigration on the growth of populations.

"The applications and uses of National Life Tables," says Dr. Farr,* "are almost innumerable: without an intimate knowledge of their properties it is impossible to determine the laws of population, which are the basis of statistics, or to reason upon such matters without falling into great errors, of which, if it were not invidious, too many instances might be cited from current works on population and public health."

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* Sixth Rep. Reg. Gen. p. 524

Table VI. — Preparatory Tables for Determining the Values of Life Annuities, and of other Single Life Benefits, at Divers Rates of Interest for Money, according to the Massachusetts Life Table.

Age.	3 per cent.		4 per cent.		5 per cent.	
	$L_x\left(\frac{1}{1\cdot03}\right)^x$	$\Sigma_x^\infty L'_x$	$L_x\left(\frac{1}{1\cdot04}\right)^x$	$\Sigma_x^\infty L'_x$	$L_x\left(\frac{1}{1\cdot05}\right)^x$	$\Sigma_x^\infty L'_x$
x	L'_x	N_x	L'_x	N_x	L'_x	N_x
0	10,000	189,437	10,000	158,231	10,000	135,595
1	8,203	179,437	8,124	148,231	8,047	125,595
2	7,289	171,234	7,150	140,107	7,014	117,548
3	6,794	163,945	6,600	132,957	6,413	110,534
4	6,449	157,151	6,204	126,357	5,971	104,121
5	6,164	150,702	5,873	120,153	5,599	98,150
10	5,115	122,031	4,643	93,311	4,220	72,992
15	4,317	98,111	3,735	72,001	3,235	53,976
20	3,564	78,061	2,938	54,969	2,426	39,491
25	2,913	61,585	2,288	41,636	1,801	28,679
30	2,368	48,151	1,772	31,277	1,330	20,671
35	1,922	37,239	1,370	23,442	980.4	14,763
40	1,557	28,388	1,058	17,069	721.4	10,410.3
45	1,256	21,228	812.8	12,291.9	528.4	7,212.3
50	1,006	15,464	620.3	8,630.9	384.4	4,875.0
55	791.5	10,876.5	465.2	5,853.5	247.8	3,184.4
60	610.6	7,291.3	342.0	3,784.4	192.6	1,984.1
65	448.8	4,570.8	239.5	2,287.9	128.6	1,156.0
70	312.5	2,609.2	158.9	1,259.6	81.33	613.51
75	199.7	1,279.4	96.75	595.36	47.20	279.41
80	99.55	490.96	45.96	219.55	21.37	99.00
85	35.44	136.02	15.59	59.44	6.911	25.73
90	8.243	25.989	3.455	10.948	1.460	4.536
95	1.236	3.363	.494	1.3177	.1988	.5210
100	.1146	.235	.0436	.0891	.0168	.0341

TABLE VI. — *continued.* PREPARATORY TABLES FOR DETERMINING THE VALUES OF LIFE ANNUITIES, AND OF OTHER SINGLE LIFE BENEFITS, AT DIVERS RATES OF INTEREST FOR MONEY, ACCORDING TO THE MASSACHUSETTS LIFE TABLE.

Age.	6 per cent.		7 per cent.		8 per cent.	
	$L_x\left(\frac{1}{1\cdot06}\right)^x$	$\Sigma_x^\infty L'_x$	$L_x\left(\frac{1}{1.07}\right)^x$	$\Sigma_x^\infty L'_x$	$L_x\left(\frac{1}{1\cdot08}\right)^x$	$\Sigma_x^\infty L'_x$
x	L'_x	N_x	L'_x	N_x	L'_x	N_x
0	10,000	118,630	10,000	105,552	10,000	95,219
1	7,971	108,630	7,896	95,552	7,823	85,219
2	6,882	100,659	6,754	87,656	6,630	77,396
3	6,234	93,777	6,060	80,902	5,894	70,766
4	5,749	87,543	5,537	74,842	5,335	64,872
5	5,340	81,795	5,095	69,305	4,863	59,537
10	3,838	58,193	3,494	47,141	3,184	38,703
15	2,806	41,207	2,438	31,948	2,120	25,099
20	2,007	28,865	1,663	21,411	1,381	16,091
25	1,421	20,077	1,124	14,256	890.7	10,250.8
30	1,001	13,870	755.1	9,433	571.3	6,493.1
35	703.6	9,502	506.5	6,194.8	365.8	4,085.1
40	493.7	6,433	339.1	4,024.2	233.8	2,544.1
45	344.9	4,283.5	226.1	2,572.9	148.7	1,560.8
50	239.3	2,784.7	149.7	1,607.3	94.00	936.60
55	163.2	1,750.7	97.36	971.6	58.37	544.23
60	109.1	1,050.2	62.08	560.91	35.53	302.20
65	69.44	589.31	37.72	303.09	20.60	157.12
70	41.89	301.41	21.71	149.33	11.32	74.545
75	23.19	132.22	11.47	63.11	5.707	30.335
80	10.01	45.00	4.724	20.62	2.245	9.524
85	3.088	11.230	1.390	4.94	.6304	2.191
90	.6221	1.894	.2672	.7973	.1157	.338
95	.0808	.2078	.0331	.0862	.01368	.0350
100	.0065	.0132	.0025	.0051	.00100	.00200

TABLE VII. — FOR DETERMINING THE AVERAGE VALUES OF LIFE ANNUITIES, AND OF OTHER SINGLE LIFE BENEFITS, ON THE WHOLE OF A STATIONARY POPULATION, OR ON THE PART AT AND OVER CERTAIN AGES, ACCORDING TO THE MASSACHUSETTS LIFE TABLE.

Interest of Money. — Five per cent.

Ages.	$Q_x \left(\frac{1}{1\cdot05}\right)^x$	$\Sigma_x^\infty \; Q'_x$	$\frac{Z_x}{Q'_x} - 1$
	Q'_x	Z_x	a'_x ANNUITY.
0	397,653	5,682,478	13.29
1	370,148		
2	345,230		
3	322,263		
4	300,885		
5	280,918	3,946,299	13.05
10	198,606	2,717,810	12.68
15	139,258	1,851,426	12.29
20	96,695	1,246,093	11.89
25	66,504	827,263	11.44
30	45,254	540,311	10.94
35	30,402	345,934	10.38
40	20,097	216,087	9.75
45	13,013	130,868	9.06
50	8,198	76,207	8.30
55	4,982	42,192	7.47
60	2,882	21,860	6.59
65	1,558	10,360	5.65
70	765	4,332	4.66
75	321	1,510	3.71
80	105.8	417.4	2.95
85	25.3	83.2	2.29
90	4.2	12.4	1.95
95	.5	1.2	1.49
100	.03	.06	1.00

The average of the present values of one dollar, payable at the close of each year during the continuance of each of the lives of the persons now at and over the age of 20 years, *in a stationary population*, interest of money being computed at the rate of 5 per cent. per annum, is $11.89. [*Table* VII.]

EXAMPLES. — The present value of one dollar, payable at the close

of each year during the continuance of the life of a person now 20 years of age, interest of money being computed at the rate of

$$\left\{\begin{array}{l}\text{4 per cent. per annum, is \$17.71}\\ \text{5 per cent. per annum, is \$15.28}\end{array}\right.$$

[*Table* VIII.]

Our L'_x is commonly written D_x.

The N_x is that used by Dr. Farr and a few other late writers, and is equivalent to the N_{x-1} introduced by Mr. Griffith Davies, and adopted in certain standard treatises on life annuities and reversions.

Q, Q', Y, and Z retain the same signification as in the Reports of the English Registrar-General.

i is any annual rate of interest for money; as, ·03, ·04, or ·05.

Formulas for determining values of annuities, annual premiums, and single premiums, are given in the headings of the respective columns in which those values appear. [Table VIII.]

$$L'_x = L_x \left(\frac{1}{1+i}\right)^x$$

$$Q'_x = Q_x \left(\frac{1}{1+i}\right)^x.$$

N_x (which equals $\Sigma_x^\infty L'_x$) $= L'_x + L'_{x+1} + L'_{x+2} + \cdots L'_\infty$, and represents the aggregate present values of a constant sum, the $\left(\frac{1}{1+i}\right)^x$ portion of one dollar, payable at the beginning of each year, during the continuance of each of the lives of the L_x persons living *at* the age x.

Z_x (which equals $\Sigma_x^\infty Q'_x$) $= Q'_x + Q'_{x+1} + Q'_{x+2} + \cdots Q'_\infty$, and represents the aggregate present values of a constant sum, the $\left(\frac{1}{1+i}\right)^x$ portion of one dollar, payable, at the beginning of each year, during the continuance of each of the lives of the Q_x persons living in a stationary population *at and over* age x.

a'_x $\left(\text{which equals } \dfrac{Z_x}{Q_x \frac{1}{(1+i)^x}} - 1\right)$ represents the average of the present values of one dollar, payable, at the beginning of each year, during the continuance of the lives of each of persons living (Q_x) in a stationary population at and over age x.

A column, represented by the well-known symbol M_x, may be constructed from values in columns L'_x and N_x by the following simple formula:

$$M_x = L'_x - \frac{i}{1+i} N_x.$$

M_x represents the aggregate present values of a constant sum, the $\frac{1}{(1+i)^x}$ portion of one dollar, payable at the end of each of the years in which the deaths of the L_x persons living at the age x will occur.

For methods for deducing from the above the values of other single life benefits, the reader is referred to the writings of Mr. David Jones, in his work on "Annuities and Reversionary Payments," of Professor De Morgan, in the Companions to the British Almanac for 1840 and 1842, and of Dr. Farr, in the Sixth and the Twelfth Reports of the Registrar-General (Eng.). These benefits may be uniform or variable, and may apply either to the entire period of life or to limited portions.

Tables, for determining the values of benefits contingent upon a combination of lives, may be framed by brief processes, in some degree analogous to those already indicated.

TABLE VIII.

Ages.	LIFE ANNUITY. — The present value, after arriving at a certain age, of one dollar, payable at the end of each year during life. $a_x = \frac{N'_x}{L'_x} - 1.$		ANNUAL PREMIUM UNAUGMENTED. — The sum payable at the beginning of each year, after arriving at a certain age, which will amount to one hundred dollars at the end of the year of decease. $\pi_x = 100\left(\frac{1}{1+a_x} - \frac{i}{1+i}\right).$		SINGLE PREMIUM UNAUGMENTED. — The present value, after arriving at a certain age, of one hundred dollars, payable at the end of the year of decease. $V_x = 100\left(1 - \frac{i}{1+i}(1+a_x)\right).$	
	4 per cent.	5 per cent.	4 per cent.	5 per cent.	4 per cent.	5 per cent.
0	14.824	12.560	2.47	2.61	39.14	35.43
1	17.247	14.608	1.63	1.65	29.82	25.68
2	18.595	15.759	1.26	1.21	24.64	20.20
3	19.145	16.236	1.12	1.04	22.52	17.92
4	19.367	16.438	1.06	.97	21.67	16.96
5	19.459	16.530	1.04	.94	21.31	16.52
10	19.097	16.297	1.13	1.02	22.70	17.64
15	18.277	15.685	1.34	1.23	25.86	20.55
20	17.710	15.278	1.50	1.38	28.04	22.48
25	17.198	14.924	1.65	1.52	30.01	24.17
30	16.651	14.542	1.82	1.67	32.11	26.99
35	16.111	14.058	2.00	1.88	34.19	28.30
40	15.133	13.431	2.35	2.17	37.95	31.28
45	14.123	12.649	2.77	2.57	41.83	35.00
50	12.914	11.682	3.34	3.12	46.49	39.61
55	11.583	10.588	4.10	3.37	51.60	44.82
60	10.065	9.301	5.19	4.95	57.44	50.95
65	8.553	7.989	6.62	6.36	63.26	57.19
70	6.927	6.543	8.77	8.50	69.51	64.08
75	5.154	4.920	12.40	12.13	76.33	71.81
80	3.777	3.633	17.09	16.82	81.63	77.94
85	2.813	2.723	22.38	22.10	85.33	82.27
90	2.169	2.101	27.71	27.50	87.81	85.24
95	1.670	1.621	33.61	33.40	89.73	87.49

TABLE IX. — COMPARISON OF THE PRESENT VALUES OF A WIDOW'S RIGHT OF DOWER IN THE INCOME OF AN ESTATE WORTH $1,000, COMPUTED ACCORDING TO THE MASSACHUSETTS, THE ENGLISH, AND THE PRUSSIAN LIFE TABLES.

Ages.	MASSACHUSETTS, 166 Towns, 1855.		ENGLAND, Females, 1841.	PRUSSIA, 1839-40-41.
	5 per cent.	4 per cent.	4 per cent.	4 per cent.
25	251	231	235	226
35	237	217	216	202
45	214	191	191	170
55	180	158	155	130
65	138	118	113	92
75	88	74	74	65
85	52	43	45	45

In computing the above table, the widow's interest in the estate was supposed to continue until the moment of decease. Such tables have been sometimes framed on the assumption that the claim was to cease with the end of the year preceding that in which the death should occur.

We observe a close resemblance between the values from the Massachusetts data, and those derived from the table that expresses the law of mortality that prevails over the females of England.

The values from Prussian data are usually less than those from the English and the American observations.

We now give tables comparing the newly determined law of mortality for Massachusetts, in some of the forms in which it has been presented, with the laws which prevail over the populations of several of the communities of Europe.

The ratios of deaths to population, in Tables X. and XI. do not, in all cases, admit of direct and exact comparison, owing to want of uniformity in the intervals of age. Their relations, however, are sufficiently obvious for our present purpose. If curves be traced, to which the ratio of the number of the living to *one* annual death, at each of the intervals of age, and the age of the middle of the interval shall be coördinates, the relative vitality of the several communities at every age of life may be readily compared, and with sufficient approach to exactness.

TABLE X. — MORTALITY, PER CENT., OR, THE NUMBER OF DEATHS TO 100 PERSONS LIVING, IN DIVERS COMMUNITIES, COMPARED.

Ages.	MASSACHUS'TS, 166 Towns, 1855.	ENGLAND AND WALES,* Seven Years, 1838-44.		SWEDEN,* Thirty Years, 1811-40.		CARLISLE,† Nine Years, 1779-87.
	Persons.	Males.	Females.	Males.	Females.	Persons.
0–5	7.32	7.07	6.04	7.28	6.27	8.23
5–10	.78	.93	.90	.83	.78	1.02
10–15	.43	.50	.55	.52	.49	.50
15–20	.88	.70	.79	.54	.53	.68
20–30	1.13	.94	.94	.90	.73	.75
30–40	1.24	1.09	1.13	1.31	1.06	1.06
40–50	1.41	1.45	1.32	1.96	1.42	1.43
50–60	2.02	2.26	1.98	3.09	2.30	1.83
60–70	3.67	4.28	3.79	5.66	4.72	4.12
70–80	7.98	9.22	8.42	11.81	10.54	8.30
80–90	18.39	20.11	18.32	25.63	23.01	17.57
90 and over	34.66	36.53	34.58	42.15	39.72	28.44
All ages	2.14	2.27	2.10	2.56	2.28	2.50

* From a paper by T. R. Edmonds, Esq., published in the numbers of "The Lancet" (London) for the 9th and the 16th of March, 1850.

† Derived from values on page 418 of Mr. Milne's Treatise on "Annuities and Assurances."

TABLE XI. — MORTALITY PER CENT., OR THE NUMBER OF DEATHS TO 100 PERSONS LIVING, IN DIVERS COMMUNITIES, COMPARED.

Ages.	BELGIUM	ENGLAND AND WALES.*		SWEDEN.†	Ages.	Ages.	PRUSSIA ‡
	9 years, 1842–50. Persons.	7 years, 1838–44. Mean of Males and Females.	10 years. 1845–54. Mean of Males and Females.	20 years, 1821–40. Mean of Males and Females.			3 years, 39,40,41. Persons.
0–1	20.11	17.92		19.84	0–1		
1–2	7.19	6.55		3.80	1–3		
2–3	3.78	3.51					
3–4	2.61	2.50		1.56	3–5		
4–5	1.80	1.84					
0–5	6.99	6.54	6.85	6.43	0–5	0–5	8.02
5–10	1.09	.91	.91	.76	5–10	5–7	1.52
10–15	.72	.53	.53	.47	10–15	7–14	.78
15–20	.87	.82	.85	.59	15–25	14–20	.63
20–25	1.04					20–25	.89
25–30	1.05	.99	1.05	.97	25–35	25–30	.97
30–35	1.08					30–35	1.08
35–40	1.21	1.25	1.30	1.42	35–45	35–40	1.32
40–45	1.44					40–45	1.45
45–50	1.56	1.66	1.76	2.06	45–55	45–55	2.10
50–55	2.08						
55–60	2.75	2.95	3.04	3.57	55–65	55–60	3.57
60–65	2.77					60–65	5.58
65–70	5.38	6.22	6.43	7.61	65–75	65–75	9.09
70–75	8.41						
75–80	11.69	13.74	14.32	16.93	75–85	75–85	15.15
80–85	16.57						
85–95	22.70	28.42	29.19	32.60	85–95	85 & upw.	26.62
95 & upw.	25.79	41.46	45.22	43.64	95 & upw.		
All ages	2.42	2.19	2.28	2.34			2.70

* Ninth Rep. Reg. Gen., p. 177, and Seventeenth Rep. Reg. Gen., p. xvi.

† Eighth Rep. Reg. Gen. (Eng.), p. 276.

‡ Proceedings Am. Assoc. for the Adv. of Science, 1856, p. 56.

Table XII. — Proportions born and surviving certain Ages in divers Communities, compared.

	Massachu'ts, 166 towns. 1855.	England and Wales, 1841. Farr.	Carlisle, 1779–87. Milne.	Prussia, 1839, 40, 41. Elliott.	Sweden and Finland, 1801–5. Milne.	Belgium, 1842–50. Elliott.
0	10,000	10,000	10,000	10,039	10,000	10,000
1	8,449	8,537	8,461	8,294	8,112	8,504
2	7,733	8,010	7,779	7,721	7,659	7,918
3	7,424	7,739	7,274	7,364	7,403	7,625
4	7,258	7,554	6,998	7,147	7,226	7,429
5	7,146	7,420	6,797	6,992	7,096	7,296
10	6,873	7,061	6,460	6,589	6,729	6,912
15	6,726	6,863	6,300	6,385	6,558	6,671
20	6,437	6,606	6,090	6,165	6,377	6,386
30	5,748	6,033	5,642	5,641	5,918	5,754
40	5,078	5,383	5,075	5,008	5,369	5,130
50	4,409	4,662	4,397	4,243	4,647	4,413
60	3,597	3,800	3,643	3,141	3,590	3,464
70	2,475	2,453	2,401	1,573	2,163	2,185
80	1,059	938	953	444	644	787
90	118	115	142	50	49	110
100	2	1	9	1	0	5

Table XIII. — Average Future Duration of Life in certain Communities, compared.

Ages.	Massachusetts.	England and Wales.			Sweden and Finland.	Prussia.	Carlisle.
	1855.	1841.		1838–44.	1801–05.	1839, 40, 41.	1779–87.
	Persons.	Males.	Females.	Males.	Persons.	Persons.	Persons.
0	39.8	40.2	42.2	40.4	39.4	36.7	38.7
5	50.2	49.6	50.4	50.2	50.0	47.1	51.3
10	47.1	47.1	47.8	47.5	47.6	44.8	48.8
15	43.0	43.4	44.1	43.6	43.8	41.2	45.0
20	39.9	39.9	40.8	40.0	40.0	37.5	41.5
25	36.9	36.5	37.5	36.6	36.3	34.0	37.9
30	34.0	33.1	34.2	33.2	32.7	30.6	34.3
35	31.0	29.8	31.0	29.8	29.1	27.1	31.0
40	27.9	26.6	27.7	26.5	25.5	23.8	27.6
45	24.6	23.3	24.4	23.1	22.1	20.4	24.5
50	21.3	20.0	21.1	19.9	18.7	17.1	21.1
55	18.1	16.7	17.6	16.7	15.6	14.0	17.6
60	15.0	13.6	14.4	13.6	12.6	11.2	14.3
65	12.1	10.9	11.5	10.9	9.9	9.0	11.8
70	9.4	8.5	9.0	8.6	7.5	7.4	9.2
75	6.8	6.6	6.9	6.6	5.7	6.0	7.0
80	5.0	4.9	5.2	5.0	4.2	4.8	5.5
85	3.7	3.7	3.8	3.7	3.2	3.8	4.1
90	2.9	2.7	2.8	2.8	2.4	3.0	3.3
95	2.3	2.0	2.1	2.1	1.7		3.5

Table XIV.—Average Future Duration of Life of a Generation, or of those Living at and over certain Ages, in a Population considered Stationary.

Ages.	Massachusetts.	England and Wales.*		
	1855.	1841.		1838–44.
	Persons.	Males.	Females.	Males.
0	32.3	31.9	32.5	32.0
10	28.6	28.2	28.7	28.2
20	24.8	24.2	24.8	24.2
30	21.0	20.3	20.9	20.3
40	17.2	16.4	17.0	16.4
50	13.4	12.7	13.2	12.7
60	9.8	9.2	9.6	9.2
70	6.5	6.3	6.5	6.3
80	4.0	4.0	4.1	4.0
90	2.6	2.4	2.5	2.5

* Sixth and Twelfth Reports Reg. Gen.

From inspection of Tables X. and XI. it appears (so far as the data show) that, from a point below age 5 to about age 15, *lower* rates of mortality obtain in Massachusetts than generally in European communities; that, from age 15 to divers ages between 35 and 50, the Massachusetts rates are much *higher;* after which they again *fall* somewhat *below* the European. Under the age of *five* years, mortality in Massachusetts seems more intense than in Europe generally, though less so than in Prussia, and less than was experienced in the town of Carlisle during the nine years, 1779–87, which period was before the introduction of vaccination.

In the first of the above-mentioned intervals (say from age 3 to age 15), the mortality of Massachusetts approaches more closely to that of Sweden than to those of the other European communities.

In the second of the intervals (from about age 17 to 45), it more nearly represents the mortality of Belgium, though *higher;* and from age 45 onwards, it is *lower* than, but nearer to, the average English rates, not varying greatly from the mortality of the females of England.

The mortality of Massachusetts appears to be *lower* than that of

Carlisle previous to about age 17, thence generally *higher* to about age 60, *lower* to about age 80, and *higher* from that point onward.

As a whole, the mortality of the State is better represented by that of England, than of any other European country.

From about age 3 to age 35 the mortality of Sweden appears to be *lower* than that of England, and after age 35, *higher*.

In Prussia, with the exception of the intervals between the ages 15 and 25, and between ages 85 and 95, the mortality is uniformly *higher* than in England.

Through much of the interval under the age of 8 years, the mortality of Belgium closely resembles that of England; from that age to about 55 it is *higher*, thence to 85 nearly the same, and above that point, *lower*.

The Belgic rate is higher than the Prussian from age 15 to 32, beyond which point it is generally the lower.

At birth, the *average future duration of life* in Massachusetts (see Table XIII.) appears to be slightly ~~less~~ greater than in Sweden. From age 5 to age 25 inclusive, it agrees well with that of the males of England, and also with that of the population of Sweden. From age 30 onwards to advanced age, it is usually best represented by that of the females of England.

For much of the period from age 30 to age 75 inclusive, the Carlisle and the Massachusetts results do not greatly differ. Our comparisons have been made with national life tables and with the Carlisle table. The latter is introduced because of its extensive employment in this country and in Europe for insurance and in legal proceedings.

We observe, according to the Massachusetts life table, that of all born alive, somewhat less than one in six (.155) die before arriving at the age of *one* year; that one fourth (.26) die before attaining the age of *three* years; that seven tenths (.71) survive the age of *five* years; one half (.51), the age of *forty* years; one fourth (.25) the age of *seventy* years; one tenth (.11), the age of *eighty* years; and that one of every hundred born alive reaches the advanced age of *ninety* years. Great reliance cannot be reposed in conclusions respecting extreme longevity derived from the data employed in the construction of any of the tables, whether European or American, both in consequence of the less reliable character of the returns at those ages, and of their limited number.

We have seen that in Massachusetts a greater disparity exists than in European countries, between the rates of mortality at the ages of 5 to 15, and the rates from age 15 to about 45.

In the towns of Massachusetts selected, the rate of mortality at the ages of 5 to 15 was but little more than one half (.55) of the rate at the ages of 15 to 40. In England, in 1841, the rate of mortality in the former interval of age was about three fourths (.78) of the rate in the latter interval.

A similar disparity is observable on comparing the returns of deaths for the entire State for the *six* years, 1850–55, with the average of the numbers living at different ages according to *two* enumerations, — the one ordered in connection with the national census for the 1st of June, 1850, and the other in connection with the State census for the 1st of June, 1855. In the six years mentioned (1850–55), the rate of mortality in the entire State, according to the returns, at the ages of 5 to 15, was fifty-six one hundredths (.56) of the rate at the ages of 15 to 40. This ratio (.56) is *almost identical* with (.55) that of the towns selected in 1855, and strengthens the conclusion, that the feature under consideration prevails in the law of mortality of the population of the State.

The returns of deaths for the six years (1850–55) probably comprise but about *eighty-five* per cent. of the actual deaths of the period.

In the foregoing pages has been presented the Life Table for Massachusetts, with divers tables deduced therefrom. Among the more important of the latter may be enumerated: tables of average future duration of life; preparatory tables for finding the values of annuities and other single life benefits, calculated at *six* different rates of interest; and tables of life annuities, annual premiums, and single premiums at two rates of interest. Tables also have been given *comparing* the rates of mortality, the proportions living at certain ages according to the Life Table, and the average future duration of life in Massachusetts, with corresponding values in several European countries.

We defer for the present a comparison of the new results with those derived from other American observations, and with those from observations respecting select classes of lives.

We append two notes, — the former (Note A) giving the formula employed in calculating the influence of immigration and emigration

on the population under the age of *five* years, preparatory to the determination of the values in the Life Table under that age; the latter (Note B) presenting the methods employed in constructing, by summary processes, from the Life Table, other tables of practical value.

Note A. — In a community unaffected by migration, and in which the births increase by a constant ratio, the following formula expresses the relation which holds between the number of births (L_0) in a given year, their annual ratio of increase $\left(\frac{1}{v}\right)$, the function which determines the number of deaths ($D_{0/x}$) under any age ($_x$) in the same year, according to the prevailing ~~invariable~~ law of mortality, supposed invariable, and the number of those living ($P_{0/5}$) under the age of five years in the middle of that year.

$$L_0 = \left\{ P_{0/5} + \int_0^5 \frac{v^{5-x} D_{0/x}}{\int_{-\frac{1}{2}}^{\frac{1}{2}} v^x} \right\} \frac{\int_{-\frac{1}{2}}^{\frac{1}{2}} v^x}{\int_0^5 v^x}.$$

Since the value of $d_x \int_{-\frac{1}{2}}^{\frac{1}{2}} v^x$ closely approximates unity, for the above formula may be substituted

$$L_0 = \frac{P_{0/5} + dx \int_0^5 v^{5-x} D_{0/x}}{dx \int_0^5 v^x}.$$

When the births are *constant*, the expression becomes

$$L_0 = \frac{P_{0/5} + dx \int_0^5 D_{0/x}}{5}.$$

This relation is more fully discussed in the Proceedings of the American Association for 1856.

$$\int_0^5 v^x\, dx = \frac{v^5 - 1}{\text{Nap. log. } v.} = \frac{v^5 - 1}{\text{Com. log. } v.} \times \cdot 4342945.$$

To obtain $\int_0^5 v^{5-x} D_{0/x}\, dx$, when v and $D_{0/1}$, $D_{0/2}$, $D_{0/3}$, $D_{0/4}$, and $D_{0/5}$ are given, first determine the values of $v^4 D_{0/1}$, $v^3 D_{0/2}$, $v^2 D_{0/3}$, and $v\, D_{0/4}$.

Then putting

S_5 for $D_{0/5} + v\, D_{0/4} + v^u D_{0/3} + v^3 D_{0/2} + v^4 D_{0/1}$,
S_4 for $v\, D_{0/4} + v^u D_{0/3} + v^3 D_{0/2} + v^4 D_{0/1}$,
S_3 for $v^2 D_{0/3} + v^3 D_{0/2} + v^4 D_{0/1}$,

and assuming an algebraic law of relation to connect the values S_2, S_4, and S_8, we have

$$\int_0^5 v^{5-x}\, D_{0/x}\, dx = \frac{S_5 + S_4}{2} - \frac{D_{0/5} - v^2 D_{0/3}}{24}.$$

Note B.—On Methods employed in the Construction of certain Tables.

$$Q_x = dx \int_x^{\infty} L_x.$$

$$Y_x = dx \int_x^{\infty} Q_x.$$

$$N_x = \Sigma_x^{\infty} L'_x = \Sigma_x^{\infty} \frac{L_x}{(1+i)^x},$$

$$Z_x = \Sigma_x^{\infty} Q'_x = \Sigma_x^{\infty} \frac{Q_x}{(1+i)^x}.$$

On account of the obvious similarity of construction of Q and Y, and also of N and Z, we need only present the methods adopted in deducing Q from L, and N from L'.

From age five onwards to advanced age, the values of L and L' are given quinquennially; from birth to age five, annually. The construction of the values in columns Q and Y, at ages earlier than five years, differs. Let S_x and S'_x represent the sum of the values of L_x and L'_x respectively, at and over any age x, at equidistant intervals of n years; that is, let

$$S_x = L_x + L_{x+n} + L_{x+2n} + \cdots$$

and

$$S'_x = L'_x + L'_{x+n} + L'_{x+2n} + \cdots$$

L_x and L'_x are general terms of series of *positive* values, that *vanish* when x is taken sufficiently great.

We remark, that

$$dx \int_x^{\infty} L_x = dx \int_x^{x+n} S_x,$$

and

$$\Sigma_x^{\infty} L'_x - \frac{L'_x}{2} = \Sigma_x^{\infty} S'_x - \frac{S'_x}{2}.$$

We then assume the following formulas of integration,

$$Q_x \left(\text{or } dx \int_x^{\infty} L_x, \text{ which equals } dx \int_x^{x+n} S_x\right),$$

equals

(A) $$n\left(\frac{S_x + S_{x+n}}{2} - \tfrac{1}{12}\cdot\frac{m\,D_x + D_{x-n}}{m+1}\right);$$

and

$$N_x\left(\text{or } \Sigma_x^{\infty} L'_x, \text{ which equals } \Sigma_x^{x+n} S'_x - \frac{S'_x}{2} + \frac{L'_x}{2}\right),$$

equals

(B) $$n\left(\frac{S'_x + S'_{x+n}}{2}\right) + \frac{L'_x}{2} - \frac{n^2-1}{12\,n}\cdot\frac{m\,D'_x + D'_{x-n}}{m+1}\Bigg),$$

in which

$$D_x = L_x - L_{x+n},$$

and

$$D'_x = L'_x - L'_{x+n};$$

Let m equal *unity*, when the ratio, $\frac{D_{x-n}}{D_x}$ or $\frac{D'_{x-n}}{D'_x}$, is *greater* than $\frac{4}{9}$, and *less* than $2\frac{1}{4}$.

$$\text{Let } m \text{ equal}\left\{\begin{array}{l}\left.\begin{array}{l}2,\\3,\\4,\\5,\end{array}\right\}\text{when the ratio is between}\left\{\begin{array}{l}2\frac{1}{4}\text{ and }3\frac{1}{2}\\3\frac{1}{2}\text{ and }5\frac{1}{2}\\5\frac{1}{2}\text{ and }7\frac{1}{2}\\7\frac{1}{2}\text{ and }9\frac{1}{2}\end{array}\right.\\\left.\begin{array}{l}5\frac{1}{2}\\5\\4\end{array}\right\}\textit{tenths of the ratio, }\text{when the ratio is between}\left\{\begin{array}{l}9\frac{1}{2}\text{ and }16\\16\text{ and }40\\40\text{ and }250\end{array}\right.\\3\ \textit{tenths of the ratio, }\text{when the ratio exceeds }250.\end{array}\right.$$

These ratios, except at quite advanced ages, will commonly be such that m will equal *unity*, and the values of Q and N will not then differ from those that result from the assumption of an *algebraic* law of relation connecting the *four* values of S_x or S'_x, at the ages $x - n$, x, $x + n$, and $x + 2n$.

If in (A) for $S_x + S_{x+n}$ the sum of the values of S_x at the limiting ages x, and $x + n$, we put G, and for $\frac{m\,D_x + D_{x-n}}{m+1}$ we put H; and in (B), in like manner, put G' and H', we shall have

(C) $$Q_x = n\left(\frac{G}{2} - \frac{H}{12}\right),$$

and

(D) $$N_x = \frac{n\,G'}{2} + \frac{L'_x}{2} - \frac{n^2-1}{12\,n}\,H'.$$

When it is desired to determine the values of Q or N from but *three* given equidistant values of S, or S', for H or H' we put the sec-

ond difference of the three values; this is equivalent to assuming that the three values are connected by an *algebraic* law of relation.

If in (C) we let H be *zero*, Q_x becomes merely the product of the average $\left(\frac{G}{2}\right)$ of the values of S_x at the limiting ages x, and $x + n$, by (n) the number of years in the interval; and is equivalent to assuming that a law of arithmetical progression connects the values of S_x within the limits.* When the interval of age is quinquennial, $\frac{n^2-1}{12n}$ equals $\frac{4}{10}$.

The operations in (C) and (D) may receive verbal interpretations.

To obtain Q_x; from the average of the limiting values $\left(\frac{S_x + S_{x+n}}{2}\right)$ of S_x, subtract one twelfth ($\frac{1}{12}$) of *a mean* (H) of the second differences (D_{x-n}, and D_x) of the four consecutive values (S_{x-n}, S_x, S_{x+n}, and S_{x+2n}) of S_x, one of which (S_{x-n}) shall precede, and another (S_{x+2n}) follow the values at the limiting ages (x and $x - n$), and multiply by the number of years (n) in the interval of age.

To obtain N_x; multiply the average of the limiting terms of S'_x by the number of years (n) in the interval of age, add one half of the value of L', corresponding to the age, and subtract $\frac{n^2-1}{n}$ twelfths of *a mean* (H') of the second differences (D'_{x-n} and D'_x) of the four values of S'_x at the ages $x - n$, x, $x + n$, and $x + 2n$.

We remark that H and H' are *arithmetical* means *only* when m equals *unity;* in other cases the greater weight is commonly given to the less of the second differences.

By giving to m the values which we have mentioned above, we are enabled readily, and without resort to logarithmic tables, to arrive at values that closely approximate those that would have resulted from the integration of the exponential function

$$a + bx + c\,d^x$$

which may be assumed to equal S_x or S'_x. a, b, c, and d are

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* This very simple form does not differ essentially from that given by Dr. Farr in the Fifth Report of the Reg. Gen. (Eng.), and is sufficiently accurate for the earlier ages, if the uniform interval of age (n) is not larger than quinquennial.

constants to be determined from the *four* values of the functions (S_x or S'_x) corresponding to the specified ages $x - n$, x, $x + n$, and $x + 2\,n$.

d will, in all cases, be *positive*, and the curve represented by above exponential function, if referred to rectangular coördinates, will have *no point of contrary flexure.*

If S_x ~~or S'_x~~ be represented by the algebraic function

$$a + bx + cx^2 + dx^3,$$

the curve, to which the above is the equation, if referred to rectangular coördinates, will have *a point of contrary flexure* within the limits of the ages $x - n$, and $x + 2n$, whenever the ratios of the second differences $\left(\frac{D_{x-n}}{D_x}\right)$ of the values of S_x corresponding to the ages $x - n$, x, $x + n$, and $x + 2\,n$ is *greater* than 2, or *less* than $\frac{1}{2}$; and the larger the ratio, if greater than 2, or the smaller the ratio, if less than $\frac{1}{2}$, the more eccentric the curve.

If in (A) we give to m the value

$$\frac{(r - 1 - 12\,\delta)\,r}{12\,\delta\,r - r + 1},$$

in which

$$r = \frac{D_{x-n}}{D_x},$$

and

$$\delta = \tfrac{1}{2}\,\frac{r+1}{r-1} - \frac{1}{\text{Nap. log.}\,r},$$

we shall obtain for Q_x precisely the values that would have resulted from the direct integration within the limits x and $x + n$, of the exponential expression,

$$S_x = a + b\,x + c\,d^x.$$

Above age 5, the values of Q_x were formed by successively adding to the previously determined value of Q_5, the values of the definite integrals of L_x for the ages 4 to 5, 3 to 4, 2 to 3, and 1 to 2, determined according to algebraic laws of relation, involving, in the first case (that from 4 to 5), *three*, and in the other cases *four* of the given equidistant values of L_x. The integral from *birth* to age 1 was determined by assuming that the values at ages 0, 1, and 2 were connected by the parabolic law of relation,

$$L_x = L_0 - (L_0 - L_1)\,x^b,$$

in which b obviously equals

$$\frac{\log.\,(L_0 - L_2) - \log.\,(L_0 - L_1)}{\log.\,2}.$$

The value of $\int_0^1 L_x\,dx$, the required integral, is

$$= \left(L_0 - \frac{L_0 - L_1}{1+b}\right).$$

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5. On a New Form of Arithmetical Complements. By Thomas Hill, of Waltham, Mass.

If we give the name of arithmetical supplement to the arithmetical complement diminished by one, or, in other words, to the complement obtained by subtracting each digit of a number, zeros included, from the highest digit of the system; (that is, in decimal notation from nine) then the following theorem is manifestly true.

If from the supplement of any whole number we subtract the same number that we add to the whole number, the sum and difference thus obtained are supplements of each other.

Thus $1863 + 857 = 2720$ and $8136 - 857 = 7279$; and 1863 is the supplement of 8136, and 2720 of 7279. These supplements may be used in arithmetical machines by printing the supplement of each digit in a smaller type by its side, so that we add by looking at the larger figures, and subtract by looking at the smaller. Thus the example already given may be printed

$$1_8 8_1 6_3 3_6 \pm 857 = 2_7 7_2 2_7 0_9.$$

Thinking that possibly other uses might be found for them, I have thus called the attention of computers to them.